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A One-Dimensional Model with Phase Transition

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A one-dimensional model is studied with nearest neighbor interaction and certain forbidden configurations. In this model it is possible to investigate the phase transition even on the microcanonical level, and it turns out that phases can coexist under certain circumstances.

KEY WORDS: Phase transition; microcanonical distribution; coexistence of phases.

1. INTRODUCTION AND SUMMARY OF RESULTS

Even the simplest existing models in which phase transition occurs are too complicated to make it possible to see in detail what happens on the microcanonical level during a phase transition. The object of the present paper is to give a model in which phase transition occurs, and which at the same time is sufficiently simple to be analyzed on the microcanonical level. I think that the model is far from realistic, but that it is justified by the above properties.

Consider a one-dimensional "crystal" with nearest neighbor interaction. The sites of the particles are identified with the integers $i = 0, 1,..., N$, and the state of the ith particle is denoted by x_i. There is only a finite set X of possible states. The interaction is determined by a pair potential $u(x, x')$, the total energy being $\sum_{i=1}^{N} u(x_{i-1}, x_i)$. The pair potential takes values in the integers extended with the point ∞. The value ∞ thus corresponds to a forbidden configuration.

The transfer matrix $Z(\theta)$ is given by

$$Z(\theta; x, y) = \begin{cases} e^{\theta u(x, y)} & \text{if} \quad u(x, y) < \infty \\ 0 & \text{if} \quad u(x, y) = \infty \end{cases} \tag{1.1}$$

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If $Z(\theta)$ is irreducible [which is the case if, for example, $u(x, y) < \infty$ for all x and y in X], then the van Hove theorem holds (see Theorem 5.6.2 in ref. 4), and hence there will be no phase transition. I therefore assume that there is a partition X_1, X_2 of X such that $\{(x, y); u(x, y) = \infty\} = X_2 \times X_1$. Then with a suitable enumeration of X, one has

$$Z(\theta) = \begin{pmatrix} Z_{11}(\theta) & Z_{12}(\theta) \\ 0 & Z_{22}(\theta) \end{pmatrix} \tag{1.2}$$

where the submatrices $Z_{11}(\theta)$ and $Z_{22}(\theta)$ are quadratic and strictly positive and hence have maximal positive eigenvalues $\lambda_1(\theta)$ and $\lambda_2(\theta)$, respectively. The matrix $Z(\theta)$ then has the maximal eigenvalue $\lambda(\theta) = \max(\lambda_1(\theta), \lambda_2(\theta))$. The eigenvalues of the submatrices are analytic in the real variable θ and hence the same is true for $\lambda(\theta)$ except at certain critical points θ for which $\lambda_1(\theta) = \lambda_2(\theta)$. These critical points are symptoms of phase transition.

In order to introduce the microcanonical distribution, I consider the energy surface $X^N(U)$ consisting of all configurations having total energy U:

$$X^N(U) = \left\{ (x_0, ..., x_N) \in X^{N+1};\ \sum_{i=1}^{N} u(x_{i-1}, x_i) = U \right\} \tag{1.3}$$

The ith particle is in phase k if $x_i \in X_k$, $k = 1, 2$. Let $T_k = T_k(x_0, ..., x_N)$ denote the number of particles in phase k. Then $T_1 + T_2 = N + 1$, and the first T_1 particles are in phase 1 and the last T_2 in phase 2.

The microcanonical distribution on the surface $X^N(U)$ assigns equal weights $1/|X^N(U)|$ to each point on the surface. Here $|X^N(U)|$ stands for the number of points in the set $X^N(U)$. This distribution thus induces a probability distribution for T_k.

It turns out (Theorems 3.1 and 4.1) that under the assumption (1.2), and if N is large, then there is a critical interval (u', u'') with the following properties:

1. If $U/N < u'$, then $T_2 = O(1)$ and the overwhelming majority of the particles will thus be in phase 1.
2. If $U/N = \alpha u' + (1 - \alpha) u''$ with $0 < \alpha < 1$, then $T_1 \approx N\alpha$ and $T_2 \approx N(1 - \alpha)$, with the dispersion being of the order $N^{1/2}$. A proportion α of the particles are thus in phase 1 and the remaining ones are in phase 2.
3. If $U/N > u''$, then $T_1 = O(1)$ and the overwhelming majority of the particles will thus be in phase 2.

In the first and last of the above cases there will be essentially only one phase, but in case 2 the two phases will coexist.

I shall also consider the corresponding canonical distribution (Theorems 3.2 and 4.2).

Another example of a one-dimensional model with phase transition can be found in ref. 3.

2. PRELIMINARIES

Let $a^N_{xy}(U)$ denote the number of configurations $(x_0,..., x_N)$ in $X^N(U)$ satisfying $x_0 = x$, $x_N = y$. Thus, in particular, $a^1_{xy}(U) = 1$ if $U = u(x, y)$, and zero otherwise. Then $|X^N(U)| = a^N_{\cdot\cdot}(U)$. Here and below a dot means summation over the corresponding variable. Thus, $a^N_{\cdot\cdot}(U) = \sum_{x \in X} \sum_{y \in X} a^N_{xy}(U)$, $Z(\theta; \cdot, y) = \sum_{x \in X} Z(\theta; x, y)$, etc.

The microcanonical distribution for T_1 equals

$$\text{prob}(T_1 = t) = \sum_{x \in X_1} \sum_{y \in X_2} a^{t-1}_{\cdot x} * a^1_{xy} * a^{N-t}_{y\cdot}(U)/a^N_{\cdot\cdot}(U), \qquad t = 1,..., N \tag{2.1}$$

Here the star denotes convolution:

$$a * b(U) = \sum_{V' + V'' = U} a(V')\, b(V'')$$

The numerator in (2.1) has to be replaced by $\sum_{y \in X_2} a^N_{y\cdot}(U)$ when $t = 0$ and by $\sum_{x \in X_1} a^N_{\cdot x}(U)$ when $t = N + 1$.

The probability that the particles M, $M + 1$,..., $M + n$ are in the states $z_0, z_1,..., z_n$ equals

$$\begin{aligned} &\text{prob}(x_M = z_0,..., x_{M+n} = z_n) \\ &\quad = a^M_{\cdot z_0} * a^1_{z_0 z_1} * \cdots * a^1_{z_{n-1} z_n} * a^{N-M-n}_{z_n \cdot}(U)/a^{(N)}_{\cdot\cdot}(U) \end{aligned} \tag{2.2}$$

The mathematical problem to be solved is thus to find approximations for quantities such as the numerators and the denominator in (2.1) and (2.2).

I shall illustrate the method by considering $a^N_{xy}(U)$ when U/N is outside the critical interval (u', u''). The quantity $a^N_{xy}(U)$ is related to $Z^N(\theta)$ in the following way:

$$\begin{aligned} Z^N(\theta; x, y) &= \sum_U e^{\theta U} a^N_{xy}(U) \\ a^N_{xy}(U) &= (2\pi)^{-1} \int_{-\pi}^{\pi} e^{-(\theta + i\xi)U} Z^N(\theta + i\xi; x, y)\, d\xi \end{aligned} \tag{2.3}$$

By induction using (1.2), we get

$$Z^N(\theta) = \begin{pmatrix} Z_{11}^N(\theta) & \sum_{i+j=N-1} Z_{11}^i(\theta)\, Z_{12}(\theta)\, Z_{22}^j(\theta) \\ 0 & Z_{22}^N(\theta) \end{pmatrix} \tag{2.4}$$

Note that

$$Z_{kk}^N(\theta) \approx \lambda_k(\theta)^N\, E_k(\theta) \tag{2.5}$$

where

$$E_k(\theta; x, y) = e_k(\theta; x)\, e_k^*(\theta; y) \tag{2.6}$$

for $x \in X_k$, $y \in X_k$, and where $e_k^*(\theta; x)$ and $e_k(\theta; x)$ are, respectively, the left and right positive eigenvectors of the submatrix $Z_{kk}(\theta)$ corresponding to the eigenvalue $\lambda_k(\theta)$. The trace of $E_k(\theta)$ equals 1. The eigenvectors can be chosen to be analytic in the real variable θ.

Let

$$\Theta_1 = \{\theta \in R; \lambda_1(\theta) > \lambda_2(\theta)\}, \qquad \Theta_2 = \{\theta \in R; \lambda_2(\theta) > \lambda_1(\theta)\} \tag{2.7}$$

Then a calculation shows that

$$Z^N(\theta) \approx \lambda(\theta)^N\, F_k(\theta) \tag{2.8}$$

when $\theta \in \Theta_k$, $k = 1, 2$. Here

$$\begin{aligned} F_1(\theta) &= \begin{pmatrix} E_1(\theta) & E_1(\theta)\, Z_{12}(\theta)[\lambda_1(\theta) - Z_{22}(\theta)]^{-1} \\ 0 & 0 \end{pmatrix} \\ F_2(\theta) &= \begin{pmatrix} 0 & [\lambda_2(\theta) - Z_{11}(\theta)]^{-1}\, Z_{12}(\theta)\, E_2(\theta) \\ 0 & E_2(\theta) \end{pmatrix} \end{aligned} \tag{2.9}$$

That is,

$$F_k(\theta; x, y) = f_k(\theta; x)\, f_k^*(\theta; y) \tag{2.10}$$

where (omitting the θ) $f_1(x)$, $f_1^*(x)$, $f_2(x)$, and f_2^* are equal to $e_1(x)$, $e_1^*(x)$, $(\lambda_2 - Z_{11})^{-1}\, Z_{12} e_2(x)$, and 0, respectively, on X_1, and to 0, $e_1^* Z_{12}(\lambda_1 - Z_{22})^{-1}\,(x)$, $e_2(x)$, and $e_2^*(x)$, respectively, on X_2.

Put

$$g_k(\theta) = \log \lambda_k(\theta), \qquad g(\theta) = \max(g_1(\theta), g_2(\theta)) \tag{2.11}$$

Assume that $\theta \in \Theta_k$, and use the abbreviations $S = [Ng''(\theta)]^{1/2}$ and $v = [U - Ng'(\theta)]/S$ in the following chain of identities and approximations:

$$\begin{aligned}
&2\pi\{\exp[\theta U - Ng(\theta)]\}\, a_{xy}^N(U) \\
&\quad = \int_{-\pi}^{\pi} \{\exp(-i\xi U)\exp[-Ng(\theta)]\}\, Z^N(\theta + i\xi; x, y)\, d\xi \\
&\quad \approx \int_{|\xi| < \delta} \exp\{-i\xi[U - Ng'(\theta)]\}\exp\{N[g(\theta + i\xi) - g(\theta) - i\xi g'(\theta)]\} \\
&\qquad \times F_k(\theta + i\xi; x, y)\, d\xi \\
&\quad = \int_{|\xi| < \delta S} \exp(-i\xi v)\exp\{N[g(\theta + i\xi/S) - g(\theta) - g'(\theta) i\xi/S]\} \\
&\qquad \times F_k(\theta + i\xi/S; x, y)\, d\xi/S \\
&\quad \approx \int_{-\infty}^{\infty} \exp(-i\xi v)\exp(-\xi^2/2)\, F_k(\theta; x, y)\, d\xi/S \\
&\quad = (2\pi)^{1/2}\exp(-v^2/2)\, F_k(\theta; x, y)/S
\end{aligned} \tag{2.12}$$

Here I used (2.8) for the first approximation and Taylor's formula

$$g(\theta + i\xi/S) - g(\theta) - g'(\theta) i\xi/S \approx g''(\theta)(i\xi/S)^2/2 \tag{2.13}$$

for the second.

These approximations can be made rigorous if $g_k''(\theta) > 0$.[1] This condition is violated if and only if there is a constant c_k and a function $w_k(x)$ such that $u(x, y) = c_k + w_k(x) - w_k(y)$.[2] If the latter is the case, one can fix a z in X_k and choose $c_k = u(z, z)$, $w_k(x) = u(x, z)$. In this case $u(x, y)$ is said to be degenerate on X_k.

There is also a normalization involved here. Put

$$v(x, y) = u(x, y) - u(z, z) - u(x, z) + u(y, z)$$

The group generated by the integers $v(x, y)$, $x \in X_k$, $y \in X_k$, is then independent of z and of the form $\{0, \pm h_k, \pm 2h_k, ...\}$ for some positive integer h_k. It was assumed above that $h_k = 1$. Otherwise, one has to multiply the expression to the right in (2.14) below by h_k.

Summing up: Assume that u is not degenerate on X_k, that $h_k = 1$, and that $\theta \in \Theta_k$. If N and U tend to ∞ in such a way that $v = [U - Ng'(\theta)]/[Ng''(\theta)]^{1/2}$ stays bounded, then

$$a_{xy}^N(U) \sim \{\exp[Ng(\theta) - \theta U]\}[2\pi N g''(\theta)]^{-1/2}[\exp(-v^2/2)]\, F_k(\theta; x, y) \tag{2.14}$$

The quantity v must be bounded, and hence this approximation can be used only when U/N belongs to or is sufficiently close to the range of g'.

The function g is convex, since it is the maximum of the two convex functions g_1 and g_2. It is easy to see that

$$g'(\Theta_1 \cup \Theta_2) = g_1'(\Theta_1) \cup g_2'(\Theta_2)$$

and that $g_1'(\Theta_1)$ and $g_2'(\Theta_2)$ are disjoint. It can also be shown that $g_k'(\Theta_k)$ is open unless $u(x, y)$ is degenerate on X_k, in whch case $g_k'(\Theta_k)$ is a one-point set.

Instead of considering the general case, I shall simply assume that the functions g_1 and g_2 cross at a point $\theta = \gamma$ and nowhere else, that $\Theta_1 = (-\infty, \gamma)$, $\Theta_2 = (\gamma, \infty)$, and that $g_1'(\gamma) < g_2'(\gamma)$. The critical interval (u', u'') mentioned in the introduction is then given by $u' = g_1'(\gamma)$, $u'' = g_2'(\gamma)$.

3. THE BEHAVIOR OUTSIDE THE CRITICAL INTERVAL

I shall here consider the probabilities (2.1) and (2.2) when U/N is outside the critical interval (u', u''). Therefore, assume that $U/N > u''$ and that $\theta \in \Theta_2$ is such that $g'(\theta) = U/N$. The case $U/N < u'$ is analogous.

Multiply the numerator in (2.1) by $e^{(\theta + i\xi)U}$, and sum over all integers U. The result is

$$Z_{11}^{t-1} Z_{12} Z_{22}^{N-t}(\theta + i\xi; \cdot, \cdot) \tag{3.1}$$

Fourier inversion now shows that the numerator in (2.1) equals

$$\int_{-\pi}^{\pi} e^{-(\theta + i\xi)U} Z_{11}^{t-1} Z_{12} Z_{22}^{N-t}(\theta + i\xi; \cdot, \cdot)\, d\xi/2\pi \tag{3.2}$$

Keep t fixed, use (2.5) with $k = 2$, and approximate (3.1) by

$$e^{(N-t)\,g(\theta + i\xi)} Z_{11}^{t-1} Z_{12} E_2(\theta + i\xi; \cdot, \cdot) \tag{3.3}$$

Proceed as in (2.12), and conclude that the numerator in (2.1) can be approximated by

$$e^{(N-t)\,g(\theta) - \theta U}[2\pi N g''(\theta)]^{-1/2}\, e^{-v^2/2} Z_{11}^{t-1} Z_{12} E_2(\theta; \cdot, \cdot) \tag{3.4}$$

Here

$$v = [U - (N-t)\,g'(\theta)]/[(N-t)\,g''(\theta)]^{1/2} = O(N^{-1/2})$$

since $U = Ng'(\theta)$. The case $t = 0$ can be treated similarly.

This and the approximation (2.14) with $v=0$ yields the following result.

Theorem 3.1. Suppose that N and U tend to ∞ in such a way that $U/N=g'(\theta)$, and that $\theta \in \Theta_2$. Then

$$\operatorname{prob}(T_1=t) \to \begin{cases} \rho & \text{for } t=0 \\ C[Z_{11}(\theta)/\lambda(\theta)]^{t-1}\, r(\cdot) & \text{for } t=1, 2,\ldots \end{cases} \tag{3.5}$$

Here

$$\begin{aligned} r(x) &= Z_{12}(\theta)\, e_2(\theta; x)/\lambda(\theta), \qquad x \in X_1 \\ \rho &= e_2(\theta; \cdot)/\{e_2(\theta; \cdot) + [1-Z_{11}(\theta)/\lambda(\theta)]^{-1}\, r(\cdot)\} \\ C &= (1-\rho)/\{[1-Z_{11}(\theta)/\lambda(\theta)]^{-1}\, r(\cdot)\} \end{aligned} \tag{3.6}$$

Note that the sum of the probabilities to the right in (3.5) equals 1, and hence that T_1 stays bounded when $N \to \infty$.

Our next object is the probability (2.2). Introduce the canonical distributions

$$p_\theta^k(x_0,\ldots, x_n) = e_k^*(\theta; x_0) \prod_{j=1}^{n} [e^{\theta u(x_{j-1}, x_j)}/\lambda_k(\theta)]\, e_k(\theta; x_n) \tag{3.7}$$

when $(x_{j-1}, x_j) \in X_k \times X_k$ for $j=1,\ldots, n$, and let $p_\theta^k(x_0,\ldots, x_n)=0$ otherwise. Then for each k there is a unique stationary Markov chain determined by these marginal densities, and the probability with respect to p_θ^k that all particles are in phase k equals 1.

The microcanonical distribution (2.2) is related to the canonical distribution (3.7) in the following way.

Theorem 3.2. Suppose that M, N, $N-M$, and U tend to ∞ in such a way that $U/N=g'(\theta)$ and that $\theta \in \Theta_k$. Then

$$\operatorname{prob}(x_M=z_0,\ldots, x_{M+n}=z_n) \to p_\theta^k(z_0,\ldots, z_n) \tag{3.8}$$

Proof. The numerator in (2.2) equals

$$\int_{-\pi}^{\pi} e^{-\omega U} Z^M(\omega; \cdot, z_0) \prod_{j=1}^{n} Z^1(\omega; z_{j-1}, z_j)\, Z^{N-M-n}(\omega; z_n, \cdot)\, d\xi/2\pi \tag{3.9}$$

Here $\omega = \theta + i\xi$. The numerator can therefore be approximated by

$$e^{(N-n)g(\theta)-\theta U}[2\pi N g''(\theta)]^{-1/2}\, F_k(\theta; \cdot, z_0)\, e^{\theta u} F_k(\theta; z_n, \cdot) \tag{3.10}$$

provided $u=u(z_0, z_1)+\cdots+u(z_{n-1}, z_n)<\infty$. The theorem now follows from the approximation (2.14) and the identity (2.10). ∎

4. THE BEHAVIOR INSIDE THE CRITICAL INTERVAL

Here I investigate the behavior of (2.1) and (2.2) when U/N belongs to the critical interval, i.e., when there is a number $0<\alpha<1$ such that $U/N = \alpha g_1'(\gamma) + (1-\alpha) g_2'(\gamma)$. Here γ is the critical point for which $g_1(\gamma) = g_2(\gamma)$.

Theorem 4.1. Suppose that N and U tend to infinity in such a way that $U/N = \alpha g_1'(\gamma) + (1-\alpha) g_2'(\gamma)$, and put

$$\tau^2 = [\alpha g_1''(\gamma) + (1-\alpha) g_2''(\gamma)]/[g_1'(\gamma) - g_2'(\gamma)]^2 \tag{4.1}$$

If $(t - N\alpha)/(N\tau^2)^{1/2} \to v$, then

$$\text{prob}(T_1 = t) \sim (2\pi N\tau^2)^{-1/2} e^{-v^2/2} \tag{4.2}$$

and hence T_1 is approximatively Gaussian with mean value $N\alpha$ and dispersion $N^{1/2}\tau$.

Proof. The expression (3.2) with $\theta = \gamma$ can be approximated by

$$\begin{aligned}\int_{-\delta}^{\delta} &\exp[-(\gamma + i\xi) U] \exp[(t-1) g_1(\gamma + i\xi) + (N-t) g_2(\gamma + i\xi)] \\ &\times E_1 Z_{12} E_2(\gamma + i\xi; \cdot, \cdot)\, d\xi/2\pi \\ &\approx \{\exp[(N-1) g(\gamma) - \gamma U]\}(2\pi s^2)^{-1/2} [\exp(-w^2/2)]\, E_1 Z_{12} E_2(\gamma; \cdot, \cdot)\end{aligned} \tag{4.3}$$

Here

$$\begin{aligned} s^2 &= (t-1) g_1''(\gamma) + (N-t) g_2''(\gamma) \sim N[\alpha g_1''(\gamma) + (1-\alpha) g_2''(\gamma)] \\ w &= [U - (t-1) g_1'(\gamma) - (N-t) g_2'(\gamma)]/s \to v \end{aligned} \tag{4.4}$$

The approximation of the denominator is slightly more delicate in this case. Start with the second identity in (2.3). Assume that $x \in X_1$ and $y \in X_2$. Use (2.4) and (2.5) and conclude

$$\begin{aligned} a_{xy}^N(U) \approx \int_{-\delta}^{\delta} &\exp[-(\gamma + i\xi) U] \sum_{j+k=N-1} \exp[j g_1(\gamma + i\xi) + k g_2(\gamma + i\xi)] \\ &\times E_1 Z_{12} E_2(\theta + i\xi; x, y)\, d\xi/2\pi \end{aligned} \tag{4.5}$$

The sum equals

$$\begin{aligned} &\frac{\exp[N g_1(\gamma + i\xi)] - \exp[N g_2(\gamma + i\xi)]}{\exp[g_1(\gamma + i\xi)] - \exp[g_2(\gamma + i\xi)]} \\ &\quad \approx \exp[(N-1) g(\gamma)] \\ &\qquad \times \frac{\exp[N(g_1'(\gamma)\, i\xi - \xi^2 g_1''(\gamma)/2)] - \exp[N(g_2'(\gamma)\, i\xi - \xi^2 g_2''(\gamma)/2)]}{i\xi[g_1'(\gamma) - g_2'(\gamma)]} \end{aligned} \tag{4.6}$$

A calculation shows that the difference between the two sides is dominated by

$$\text{const} \cdot e^{(N-1)g(\gamma)}(1+N\xi^2)\, e^{-cN\xi^2} \tag{4.7}$$

for $|\xi| < \delta$, provided δ is sufficiently small. Here c is a positive constant.

Replace the sum in (4.5) by the approximation to the right in (4.6). The result is

$$e^{(N-1)g(\gamma)-\gamma U}[\Phi(v_1)-\Phi(v_2)][g_2'(\gamma)-g_1'(\gamma)]^{-1}\, E_1 Z_{12} E_2(\gamma; x, y) \tag{4.8}$$

Here

$$\begin{aligned} v_j &= [U - Ng_j'(\gamma)][Ng_j''(\gamma)]^{-1/2}, \qquad j=1,2 \\ \Phi(v) &= \int_{-\infty}^{v} (2\pi)^{-1/2} \exp(-t^2/2)\, dt \end{aligned} \tag{4.9}$$

This is so because

$$\begin{aligned} &(2\pi)^{-1} \int_{-\infty}^{\infty} [\exp(-i\xi U)][\exp(i\xi a - \xi^2 A^2/2) - \exp(i\xi b - \xi^2 B^2/2)]\, d\xi/(-i\xi) \\ &\quad = \Phi((U-a)/A) - \Phi((U-b)/B) \end{aligned} \tag{4.10}$$

Note that $v_1 \to \infty$, $v_2 \to -\infty$, and hence that $\Phi(v_1) - \Phi(v_2) \to 1$. It follows now from (4.7) that the error in the approximation is of smaller order than (4.8).

We have thus found an approximation for $a_{xy}^N(U)$ when $x \in X_1$ and $y \in X_2$. These quantities are of smaller order of magnitude when both x and y belong to the same X_k, and hence

$$a_{\cdot\cdot}^N(U) \sim e^{(N-1)g(\gamma)-\gamma U}[g_2'(\gamma)-g_1'(\gamma)]^{-1}\, E_1 Z_{12} E_2(\gamma; \cdot, \cdot) \tag{4.11}$$

The probability to the left in (4.2) behaves asymptotically as the quotient between the right-hand sides of (4.3) and (4.11). ∎

Now consider the expression (2.2). It is a consequence of Theorem 4.1 that the probability that a phase transition occurs among the particles M, $M+1,\ldots, M+n$ tends to zero, and the probability that it occurs before M tends to $\Phi(v)$ as $(M - N\alpha)/(N\tau^2)^{1/2} \to v$.

Theorem 4.2. Suppose that M, N, and U tend to ∞ in such a way that $U/N = \alpha g_1'(\gamma) + (1-\alpha) g_2'(\gamma)$, and $(M - N\alpha)/(N\tau^2)^{1/2} \to v$. Then

$$\operatorname{prob}(x_M = z_0, ..., x_{M+n} = z_n) \to (1-\beta) p_\gamma^1(z_0, ..., z_n) + \beta p_\gamma^2(z_0, ..., z_n) \tag{4.12}$$

Here $\beta = \Phi(v)$ and τ is as in (4.1).

It is possible but hardly desirable to give a detailed proof similar to the proof of Theorem 4.1.

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